Theory of Parcel Helicity in Tornadic Supercells

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ABSTRACT

Parcel helicity, h, is the scalar product of a fluid parcel's storm-relative velocity, $\mathbf{v}$ and its dilatation times vorticity, $\mathbf{w}$. Parcels that enter tornadoes possess helicities hundred times larger than environmental values. Theory is used to investigate large parcel helicities and why storm-relative environmental helicity, despite its relative smallness, might be a useful tornado forecast parameter. To obtain tractable mathematics, the flow is specified to be dry, frictionless, and isentropic. A Lagrangian integral of the equation of motion results in formulas for the covariant wind components of parcel wind and thence $\mathbf{w}$ and parcel helicity.

The $\mathbf{w}$-vector partitions into barotropic and baroclinic parts ($\mathbf{w}_{BT}$ and $\mathbf{w}_{BC}$), and velocity decomposes into three parts, a potential part ($\mathbf{v}_{\Phi}=\nabla\Phi$) and parts ($\mathbf{v}_{BT}$ and $\mathbf{v}_{BC}$) induced by barotropic and baroclinic vorticities. Here Φ is the Lagrangian integral of parcel kinetic energy minus static energy. When the environment is horizontally uniform, it is shown that $h = \mathbf{w}\bullet\nabla(\Phi+\phi_0)$ where ϕ_0 is the potential of the environmental wind. Thus, only the irrotational wind $\mathbf{v}_{\phi} \equiv \nabla(\Phi+\phi_0)$ contributes to parcel helicity.

The $\mathbf{v}_{\phi}$-wind is important because it describes left-turning, subsiding, potential flow from the environment around the storm's warm-core rotating updraft and into the tornado. As a parcel turns leftward and sinks, its overall vorticity is turned more streamwise by the river-bend effect. Simultaneous vortex stretching and amplification of Φ in vortex-sink flow accounts for large helicity in the tornado.

SIGNIFICANCE STATEMENT

Understanding environmental conditions that favor development of quasi-steady and particularly dangerous supercells is vital to public safety as these supercells often produce a major outbreak of long-lived, long-track, violent tornadoes. During its passage from the environment to a strong tornado, the helicity (scalar product of vorticity and storm-relative wind) of a parcel increases by a hundred fold or more. An analytical formula developed herein further tornadogenesis knowledge by demonstrating how such immense parcel helicities can arise. The skill of storm-relative environmental helicity as a tornado forecast parameter is apparently based on its ability to predict the storm-scale rotation that precedes tornadogenesis.

1. Introduction

Tornadoes form very rapidly through a mechanism that still is not completely understood. Recent reviews of tornadoes include Rotunno (2013), Davies-Jones (2015a), Fischer et al. (2024) and Rotunno and Bluestein (2024). Tornado theoreticians often address the problem of how a tornado acquires enormous vertical vorticity. An alternative question tackled in this paper is how tornadoes obtain enormous parcel helicities. To make inroads into how tornadoes form and sustain themselves, the theory herein mostly assumes dry, isentropic, and inviscid flow in a sheared, unstably stratified, horizontally uniform environment. An analytical formula is obtained that describes how the helicity of a parcel changes in a simplified flow that contains the essential ingredients for supercellular convection.

Important quantities in supercell dynamics are streamwise vorticity, ω_s, which is the component of vorticity in the direction of the storm-relative 3D wind, and a new quantity, parcel helicity, h, which is dilatation times the scalar product of parcel storm-relative 3D velocity, $\mathbf{v}$, and vorticity, $\boldsymbol{\omega}$. [Dilatation is the ratio of a parcel's current volume to its initial volume. For small vertical displacements, it is nearly one.] Equivalently, parcel helicity is dilatation times streamwise vorticity times the storm-relative wind speed, q.

In frictionless flow, there are two types of vorticities, barotropic and baroclinic (Dutton 1976, pp. 385-390). In the context of this paper, barotropic vorticity is vorticity present in the far-upstream environment that is subsequently stretched and reoriented by being frozen in the fluid. In contrast, baroclinic vorticity is absent far upstream. It is generated by solenoids and then stretched and realigned.

The rotating updraft of a supercell is due to tilting of environmental streamwise vorticity (Davies-Jones 1984, hereafter DJ84). It superficially resembles an axisymmetric Beltrami flow (BF) in an environment with purely streamwise vorticity (Davies-Jones 2008). A BF is a flow in which $\boldsymbol{\omega}$ is always parallel to $\mathbf{v}$. In a BF, the Bernoulli function (the sum of parcel kinetic and static energies) is spatially and temporally constant, and the parcel helicity varies as q^2. At mid-levels in a typical mesocyclone, the parcel helicity is comparable to its environmental value so amplification and reorientation of low-level environmental streamwise vorticity as in a BF can explain the updraft's rotation. In nature, a low-precipitation (LP) supercell (Davies-Jones et al. 1976, Bluestein and Parks 1983) is a prime example of a rotating updraft that resembles a BF. Its updraft is not loaded with water because the rain and hail that forms within it exhausts into the anvil. The base of the cumulonimbus is virtually free of precipitation. Despite their strong mid-level rotation and frequent production of funnel clouds, LP supercells seldom produce significant tornadoes.

Even though low-level environmental helicity is insignificant compared to the helicity in a tornado, it is paradoxically still an important predictor of tornadic supercells (MR14). The skill exhibited storm-relative environmental helicity as a tornado-forecast parameter (Markowski and Richardson 2014, hereafter MR14) must be due to its association with environmental streamwise vorticity, which is the origin of updraft rotation (DJ84).

A tornado is far from a BF. In an incompressible BF, the abnormality λ is defined by $\boldsymbol{\omega} = \lambda\mathbf{v}$ and is a universal constant. In real environments, we herein define λ as the dilatation times ω_s divided by q. Parcels that enter significant tornadoes have huge helicities. For example, in the violent Raleigh, NC tornado of 28 November 1988, the average parcel helicity, wind speed, and streamwise vorticity in the lowest 1 km of the environment were about 0.33 m s^{-2}, 16.7 m s^{-1}, and 0.02 s^{-1} (as deduced from fig. 5 in Davies-Jones 2021). If the maximum wind and vorticity in the tornado were roughly 100 m s^{-1} and 2 s^{-1}, the helicity of a parcel entering the tornado was roughly 200 m s^{-2} or 600 times the environmental value even though the windspeed only increased sixfold. If the flow were a Beltrami flow, the parcel helicity would have increased only 36 times. For the low-level Raleigh environment, the estimated abnormality is 0.0012 m^{-1}. Based on the above figures, the ratio of λ in the tornado to that in the environment was about 16 in contrast to one for a BF.

High amplifications of parcel helicity and streamwise vorticity occur in numerical simulations of supercells such as MR14's idealized supercell simulation with a free-slip lower boundary condition. This simulation produced a tornado-like vortex (TLV). Using inviscid vorticity dynamics with Beltrami type behavior, Rotunno et al. (2017) explained the vorticity of a parcel on its 'final approach' to the TLV. The parcel descended to just two meters above the ground, before rising and entering the TLV. From the numerical values provided, we can calculate the amplification factors. At the nadir, the parcel's helicity, streamwise vorticity, and λ are in turn 130, 50, and 20 times their environmental values. Since λ is conserved in a Beltrami flow, the increase in λ must have occurred before the final approach and so it is not due to frictional interaction of the flow with the ground. Instead, the growth in λ must have occurred earlier when the parcel was at higher altitude.

In contrast to the rotating supercell updraft, which originates from upward tilting and stretching of streamwise barotropic vorticity (DJ84), the parcels that enter a tornado may have descended (Davies-Jones 1982) with their horizontal vorticity greatly enhanced baroclinically during their shallow descent (e.g., Rotunno and Klemp 1985; Davies-Jones and Brooks 1993; Wicker and Wilhelmson 1995; Adlerman et al., 1999; Rotunno et al. 2017; Davies-Jones 2017, hereafter DJ17; Davies-Jones 2022, hereafter DJ22). Rotating rain curtains form in the updraft and give rise within the outer part of the mesocyclone to left-turning subsiding flow driven by inertial and downward buoyancy forces. A 'river-bend' effect occurs whereby positive transverse vorticity is turned into streamwise vorticity (Adlerman et al. 1999; Davies-Jones et al. 2001; MR14; DJ17, DJ22). The vorticity of a parcel near the ground in a mesocyclone is predominantly baroclinic according to Rotunno and Klemp (1985), Dahl et al. (2014) and Dahl (2015). The large vertical vorticity in the tornado is due to an intense upward pressure-gradient force that tips horizontal vortex lines abruptly upward. Dahl and Fischer (2023) investigated the origins of rotation for 7 parcels that entered the base of a tornado-like vortex in a simulated supercell. Although surface drag, baroclinic vorticity and preexisting vorticity all contributed to the rotation of *some* of these parcels, their results indicated no universal source of rotation.

Previous work (DJ17; DJ22) derived explicit formulas for parcel vorticity and streamwise vorticity. In the special case of, inviscid, isentropic flow, a formula for parcel helicity is developed herein. This special case is important for understanding quasi-steady supercells and as a first step to unravelling the mysteries of tornadogenesis. The Lagrangian approach utilizes the contravariant and covariant components of vectors (Fleisch 2012; Dahl et al. 2014). Section 2 provides the governing equations and section 3 derives the equations that govern parcel helicity, λ and streamwise vorticity in a Eulerian framework. Section 4 reviews non-orthogonal coordinate systems for the reader's benefit. In section 5 the covariant wind components are discovered and then used to form the scalar product of wind and vorticity and hence to obtain the formulas for parcel helicity and λ. The math perhaps could be simplified by using differential forms (Collier 2021). Section 6 explores the case where the environment is horizontally uniform and unchanging. Section 7 demonstrates that for steady isentropic flow with zero PV the formula for λ is the integral of Scorer's (1997) equation for streamwise vorticity. It also reveals the connection between curved motion and helicity production. Conclusions are listed in section 8.

Subsequent papers (Davies-Jones 2025a, b) illustrative helicity fields calculated from idealized models. Davies-Jones (2025a) uses the Davies-Jones (2000) analytical model that adopts a primary-flow, secondary-flow approach. The primary flow is potential flow around a sphere. Barotropic and baroclinic vorticities are computed as secondary effects. Davies-Jones (2025b) presents helicity calculations in the Davies-Jones (2008) axisymmetric numerical model of tornadogenesis.

2. Governing equations

In a non-rotating reference frame translating with a storm, the equations governing the motion, mass conservation, and entropy are

$$\frac{D\mathbf{v}}{Dt} = -c_p\theta\nabla\pi - \nabla(gz) + \mathbf{G}, \tag{2.1}$$

$$\frac{D}{Dt}\frac{\alpha}{\alpha_0} = \frac{\alpha}{\alpha_0}\nabla\cdot\mathbf{v}, \tag{2.2}$$

$$\frac{D\theta}{Dt} = \frac{1}{c_p\pi}\frac{\delta H}{\delta t}, \tag{2.3}$$

where

$$\ln\theta = (c_v/R)\ln\pi + \ln\alpha + \text{constant}. \tag{2.4}$$

Here, t is time, D/Dt is the material derivative, $\mathbf{v}$ is the storm-relative wind vector, q is storm-relative windspeed, θ is potential temperature, , g (= 9.80 m s^{-2}) is the acceleration due to gravity, z is height above ground, q_L is the rain mixing ratio, $\mathbf{G} \equiv \mathbf{F} - gq_L\mathbf{k}$ where $\mathbf{F}$ is friction and $-gq_L\mathbf{k}$ is the hydrometeor drag, α_0 is the initial value of a parcel's specific volume α, α/α_0 is the dilatation, c_p (=1006 m^2 s^{-2} K^{-1}) and c_v (= 719 m^2 s^{-2} K^{-1}) are the specific heats of dry air at constant pressure and volume, R (= $c_p - c_v$) is the gas constant for dry air, $\kappa = R/c_p$, p_0 = 1000 mb, $\pi = (p/p_0)^\kappa$ is nondimensional pressure, and $\delta H/\delta t$ is the rate that heat is added to a parcel per unit mass. All quantities in this paper are storm relative. When $\mathbf{G} \equiv 0$, we dub the motion 'dragless'. Equations (2.1)-(2.4) represent a closed system in the variables $\mathbf{v}$, π, θ and α. Related variables are temperature T, specific entropy S, the specific static energy σ, and the Bernoulli function B. These are defined by

$$T \equiv \pi\theta, \tag{2.5}$$

$$S \equiv c_p\ln\theta + \text{constant}, \tag{2.6}$$

$$\sigma \equiv c_pT + gz, \tag{2.7}$$

$$B \equiv \sigma + q^2/2. \tag{2.8}$$

Where the static energy is the sum of enthalpy and potential energy. The vorticity is

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}. \tag{2.9}$$

For convenience we define the vector

$$\mathbf{w} \equiv \frac{\alpha}{\alpha_0}\boldsymbol{\omega} \tag{2.10}$$

(not to be confused with the scalar w, which is used for vertical velocity).

Using (2.5 and 2.7) and a vector identity we can rewrite (2.1) as

$$\frac{D\mathbf{v}}{Dt} \equiv \frac{\partial \mathbf{v}}{\partial \tau} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \frac{q^2}{2} - \mathbf{v} \times \boldsymbol{\omega} = -\nabla\sigma + \pi c_p \nabla\theta + \mathbf{G}, \tag{2.11}$$

where $\mathbf{v}\times\boldsymbol{\omega}$ is the Lamb vector. Taking the curl of (2.1) or (2.11) and utilizing (2.2) yields the vorticity equation

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w} \cdot \nabla)\mathbf{v} + \frac{\alpha}{\alpha_0}\left(\nabla\pi \times c_p \nabla\theta + \nabla \times \mathbf{G}\right) \tag{2.12}$$

where $\nabla\times\mathbf{G} \equiv \nabla\times\mathbf{F} - g\nabla q_L \times \mathbf{k}$. In the special case of Boussinesq flow with Fickian diffusion, the divergence of (2.11) yields a diagnostic equation for B (Adrian 1982), namely

$$-\nabla \bullet \nabla(B + g q_L \mathbf{k}) = -\nabla \bullet (\mathbf{v} \times \boldsymbol{\omega}). \tag{2.13}$$

For future use, we make the following definitions. The unit vectors **i**, **j**, **k** form a Cartesian orthonormal set of basis vectors with **i** eastward, **j** northward, and **k** vertically upward. The position vector and wind in these Eulerian coordinates are $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. The Lagrangian coordinates of a parcel are (X, Y, Z, τ) or (X^1, X^2, X^3, τ) in tensor notation where τ is the time and $\mathbf{X} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ is the parcel's position vector at the initial time τ_0 (or t_0). Subscript 0 denotes initial quantities. The material derivative is D/Dt in Eulerian coordinates and $\partial/\partial\tau$ in Lagrangian coordinates.

3. Formula for the helicity of moving parcels

We now derive the governing equation for the parcel helicity $h \equiv (\alpha/\alpha_0)\boldsymbol{\omega}\bullet\mathbf{v} = \mathbf{w}\bullet\mathbf{v}$. Via the product rule for derivatives,

$$\frac{Dh}{Dt} \equiv \frac{D}{Dt}(\mathbf{w}\cdot\mathbf{v}) = \mathbf{w}\cdot\frac{D\mathbf{v}}{Dt} + \mathbf{v}\cdot\frac{D\mathbf{w}}{Dt}. \tag{3.1}$$

Introducing the vorticity equation (2.12) and the equation of motion (2.11) produces the parcel-helicity equation,

$$\frac{Dh}{Dt} = \mathbf{v}\cdot(\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{w}\cdot\left(-\nabla\sigma + \pi c_p\nabla\theta + \mathbf{G}\right) + \mathbf{v}\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right)$$

$$= \mathbf{w}\cdot\nabla\left(\frac{q^2}{2} - \sigma\right) + \mathbf{w}\cdot\left(\pi c_p\nabla\theta + \mathbf{G}\right) + \frac{\alpha}{\alpha_0}\mathbf{v}\cdot\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right) \tag{3.2}$$

where we have used a vector identity. Lilly (1986) derived a less general version of (3.2).

When the motion is dragless ($\mathbf{G} \equiv \mathbf{0}$) and isentropic ($D\theta/Dt \equiv 0$), (3.2) has an integral that is derived here via a partly Eulerian proof. A more direct Lagrangian derivation is presented in section 5. We prove in appendix A that the potential vorticity (PV), of a parcel in isentropic motion is conserved. To proceed, we need some definitions. Let

$$\begin{gathered} D\Phi/Dt = q^2/2 - \sigma = q^2 - B, \\ \Phi = 0 \text{ at } t = t_0. \end{gathered} \tag{3.3}$$

where $D\Phi/Dt$ is the parcel's specific kinetic energy minus its static energy or, alternatively, the parcel's speed squared minus its Bernoulli function, $B \equiv q^2/2 + \sigma$. Let Π be the cumulative (time-integrated) non-dimensional pressure where

$$D\Pi/Dt = \pi, \Pi = 0 \text{ at } t = t_0. \tag{3.4}$$

We can compute Φ and Π by integrating (3.3) and (3.4) along characteristics (i.e., trajectories). We also define

$$\begin{gathered} \mathbf{v}_{\mathrm{BT}} \equiv u_0\nabla X + v_0\nabla Y + w_0\nabla Z, \\ \mathbf{v}_\Phi \equiv \nabla\Phi,\ \mathbf{v}_{\mathrm{BC}} \equiv \Pi c_p\nabla\theta. \end{gathered} \tag{3.5}$$

By applying the identity (A5) to (3.5), and using (3.3) and (3.4), we get

$$\frac{D}{Dt}(\mathbf{w}\cdot\mathbf{v}_\Phi) \equiv \frac{D}{Dt}(\mathbf{w}\cdot\nabla\Phi) = \mathbf{w}\cdot\nabla\left(\frac{q^2}{2} - \sigma\right) + \mathbf{v}_\Phi\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right), \tag{3.6}$$

$$\frac{D}{Dt}(\mathbf{w}\cdot\mathbf{v}_{\mathrm{BT}}) \equiv \frac{D}{Dt}\left[\mathbf{w}\cdot(u_0\nabla X + v_0\nabla Y + w_0\nabla Z)\right]$$

$$= \mathbf{v}_{\mathrm{BT}}\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right), \tag{3.7}$$

$$\frac{D}{Dt}(\mathbf{w}\cdot\mathbf{v}_{\mathrm{BC}}) \equiv \frac{D}{Dt}\left(\mathbf{w}\cdot\Pi c_p\nabla\theta\right) = \mathbf{w}\cdot\left(\pi c_p\nabla\theta + \Pi c_p\nabla\frac{D\theta}{Dt}\right)$$
$$+\mathbf{v}_{\mathrm{BC}}\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right), \tag{3.8}$$

since X, Y, Z, u_0, v_0 and w_0 are constants of the motion. Subtracting (3.6), (3.7) and (3.8) from (3.2) yields

$$\frac{D}{Dt}[h - \mathbf{w}\cdot(\mathbf{v}_\Phi + \mathbf{v}_{\mathrm{BT}} + \mathbf{v}_{\mathrm{BC}})] = \mathbf{w}\cdot\left(\mathbf{G} - \Pi c_p\nabla\frac{D\theta}{Dt}\right)$$

$$+(\mathbf{v} - \mathbf{v}_\Phi - \mathbf{v}_{\mathrm{BT}} - \mathbf{v}_{\mathrm{BC}})\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right). \tag{3.9}$$

To obtain an integral of the helicity equation, we assume that a parcel of interest, P, and its surroundings are in dry isentropic frictionless motion, so

$$\theta = \theta(X, Y, Z) \tag{3.10}$$

and $\mathbf{G} = \mathbf{0}$. There is no need to stipulate that the entire flow be isentropic and frictionless. In this case, Dutton (1976, p. 388), Mobbs (1981, 1982), Davies-Jones (2015b) and DJ22 showed that

$$\mathbf{v} = \mathbf{v}_{\mathrm{BT}} + \mathbf{v}_\Phi + \mathbf{v}_{\mathrm{BC}}. \tag{3.11}$$

In this decomposition of $\mathbf{v}$, which we prove in section 5 for the reader's benefit, $\mathbf{v}_{\mathrm{BT}}$ is a 'barotropic velocity' that depends on initial conditions, $\mathbf{v}_\Phi$ is an irrotational Φ-dependent velocity, and $\mathbf{v}_{\mathrm{BC}}$ is a 'baroclinic velocity' that depends on entropy gradient and accumulated pressure. The latter two velocities are both zero initially. From the curls of (3.5) and (3.11), $\boldsymbol{\omega}$ decomposes into barotropic vorticity, $\boldsymbol{\omega}_{\mathrm{BT}}$, and baroclinic vorticity, $\boldsymbol{\omega}_{\mathrm{BC}}$. Specifically,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{BT}} + \boldsymbol{\omega}_{\mathrm{BC}},$$
$$\boldsymbol{\omega}_{\mathrm{BT}} = \nabla u_0\times\nabla X + \nabla v_0\times\nabla Y + \nabla w_0\times\nabla Z,$$
$$\boldsymbol{\omega}_{\mathrm{BC}} = \nabla\Pi\times c_p\nabla\theta. \tag{3.12}$$

By (3.12) the baroclinic vortex lines are tangent to the isentropic surfaces. In aerodynamical language (Milne-Thompson 1973, pp. 167-168), $\mathbf{v}_{\mathrm{BT}}$ and $\mathbf{v}_{\mathrm{BC}}$ are the velocities induced (*not caused*) by $\boldsymbol{\omega}_{\mathrm{BT}}$ and $\boldsymbol{\omega}_{\mathrm{BC}}$, respectively, while $\mathbf{v}_\Phi$ is the residual irrotational flow (not induced by any vorticity). We also define $\mathbf{w}_{\mathrm{BT}} \equiv (\alpha/\alpha_0)\boldsymbol{\omega}_{\mathrm{BT}}$ and $\mathbf{w}_{\mathrm{BC}} \equiv (\alpha/\alpha_0)\boldsymbol{\omega}_{\mathrm{BC}}$.

From (3.5) and (3.12), the helicity of a parcel in dry inviscid isentropic motion is

$$h = (\mathbf{w}_{\mathrm{BT}} + \mathbf{w}_{\mathrm{BC}}) \bullet (\mathbf{v}_{\mathrm{BT}} + \mathbf{v}_\Phi + \mathbf{v}_{\mathrm{BC}}) \tag{3.13}$$

Initially $\mathbf{w}_{\mathrm{BC}}$, $\mathbf{v}_{\Phi}$ and $\mathbf{v}_{\mathrm{BC}}$ are zero, so the initial helicity

$$h_0 = (\mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\mathrm{BT}})_0. \tag{3.14}$$

From (3.5) and (3.12) and use of Jacobian notation (Hildebrand, 1962, p. 344),

$$\mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\mathrm{BT}} = \frac{\alpha}{\alpha_0}(\nabla u_0 \times \nabla X + \nabla v_0 \times \nabla Y + \nabla w_0 \times \nabla Z) \bullet (u_0 \nabla X + v_0 \nabla Y + w_0 \nabla Z)$$

$$= \frac{\alpha}{\alpha_0} u_0 \left[\frac{\partial(w_0, Z, X)}{\partial(x,y,z)} - \frac{\partial(v_0, X, Y)}{\partial(x,y,z)}\right] + \frac{\alpha}{\alpha_0} v_0 \left[\frac{\partial(u_0, X, Y)}{\partial(x,y,z)} - \frac{\partial(w_0, Y, Z)}{\partial(x,y,z)}\right]$$
$$+ \frac{\alpha}{\alpha_0} w_0 \left[\frac{\partial(v_0, Y, Z)}{\partial(x,y,z)} - \frac{\partial(u_0, Z, X)}{\partial(x,y,z)}\right]. \tag{3.15}$$

The Lagrangian continuity equation is

$$\frac{\alpha}{\alpha_0} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)}. \tag{3.16}$$

from Salmon (1998, p. 6). By (3.16) and properties of the Jacobian,

$$\frac{\alpha}{\alpha_0}\frac{\partial(w_0, Z, X)}{\partial(x,y,z)} = \frac{\partial(w_0, Z, X)}{\partial(x,y,z)}\frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \frac{\partial(w_0, Z, X)}{\partial(X,Y,Z)}, \text{etc.} \tag{3.17}$$

After applying (3.17) to (3.15), we obtain

$$\mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\mathrm{BT}} = u_0\left(\frac{\partial w_0}{\partial Y} - \frac{\partial v_0}{\partial Z}\right) + v_0\left(\frac{\partial u_0}{\partial Z} - \frac{\partial w_0}{\partial X}\right) + w_0\left(\frac{\partial v_0}{\partial X} - \frac{\partial u_0}{\partial Y}\right) = h_0. \tag{3.18}$$

Thus,

$$\mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\mathrm{BT}} = h_0. \tag{3.19}$$

Other terms in (3.13) also simplify. From (3.5) and (3.12),

$$\mathbf{w}_{\mathrm{BC}} \cdot \mathbf{v}_{\mathrm{BC}} = \frac{\alpha}{\alpha_0} \nabla\Pi \times c_p \nabla\theta \cdot \Pi c_p \nabla\theta = 0, \tag{3.20}$$

which states that the baroclinic velocity is normal to the baroclinic vorticity. This is exemplified by a density current that propagates perpendicular to its roll cloud. With (3.19) and (3.20), (3.13) becomes

$$h = h_0 + \mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\Phi} + \mathbf{w}_{\mathrm{BC}} \bullet \mathbf{v}_{\Phi} + \mathbf{w}_{\mathrm{BC}} \bullet \mathbf{v}_{\mathrm{BT}} + \mathbf{w}_{\mathrm{BT}} \bullet \mathbf{v}_{\mathrm{BC}}. \tag{3.21}$$

If the flow under consideration has zero PV,

$$\mathbf{w} \cdot \mathbf{v}_{\mathrm{BC}} = \Pi c_p \mathbf{w} \cdot \nabla\theta = 0 \tag{3.22}$$

from (3.5), and, by subtraction of (3.20) from (3.22),

$$\mathbf{w}_{\mathrm{BT}} \cdot \mathbf{v}_{\mathrm{BC}} = \Pi c_p \mathbf{w}_{\mathrm{BT}} \cdot \nabla\theta = 0. \tag{3.23}$$

Therefore, by (3.20) and (3.23), the barotropic and the baroclinic vortex lines both lie in the isentropic surfaces of a zero PV flow. For inviscid isentropic motion with zero PV, (3.21) reduces to

$$h = h_0 + \mathbf{w}_{\mathrm{BT}} \bullet \nabla\Phi + \mathbf{w}_{\mathrm{BC}} \bullet \mathbf{v} \tag{3.24}$$

via (3.19), (3.20), (3.23), (3.11), and (3.5). For inviscid, homentropic (constant θ) motion, (3.24) is simply

$$h - \mathbf{w}_{\mathrm{BT}} \bullet \nabla\Phi = h_0 \tag{3.25}$$

so $h - \mathbf{w}_{\mathrm{BT}} \bullet \nabla\Phi$ is a constant of this motion. Appendix B verifies (3.25) for an unsteady Rankine combined vortex.

In summary, (3.21) is the integral of (3.2) for parcels in dry, frictionless, isentropic motion. It suggests an important role for $\mathbf{v}_\Phi \equiv \nabla\Phi$ in tornadogenesis that is invisible to the customary supercell diagnostics that trace the evolution of material circulation (e.g., Rotunno and Klemp 1985, Davies-Jones and Brooks 1993). Circulation analyses fail to uncover its significance because the circulation around a closed circuit is unaffected by any path-independent vector field (Davies-Jones and Markowski 2021).

4. Review of non-orthogonal coordinate systems

We briefly review some properties of nonorthogonal coordinates (Margenau and Murphy 1956, pp. 192-197) as they relate to Lagrangian coordinates in a fluid dynamical context. The Jacobian matrix of the transformation $\mathbf{T} \equiv \mathbf{x}(\mathbf{X}, \tau)$ from Lagrangian to Eulerian coordinates is

$$\mathbf{J} \equiv \begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix} \tag{4.1}$$

and the inverse transformation is

$$\mathbf{J}^{-1} \equiv \begin{bmatrix} \partial X/\partial x & \partial X/\partial y & \partial X/\partial z \\ \partial Y/\partial x & \partial Y/\partial y & \partial Y/\partial z \\ \partial Z/\partial x & \partial Z/\partial y & \partial Z/\partial z \end{bmatrix}. \tag{4.2}$$

The column vectors of $\mathbf{J}$ are the covariant basis vectors $\mathbf{e}_1 = \partial\mathbf{x}/\partial X$, $\mathbf{e}_2 = \partial\mathbf{x}/\partial Y$, $\mathbf{e}_3 = \partial\mathbf{x}/\partial Z$. These are tangential to the X-, Y- and Z-coordinate curves. A parcel's $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors are tangent to its Z-surface while its $\mathbf{e}_3$ vector is initially normal to it and thereafter generally oblique to it. The contravariant basis vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, are reciprocal to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and vice versa so

$$\mathbf{e}^1 = \mathbf{e}_2 \times \mathbf{e}_3/\det\mathbf{J}, \mathbf{e}^2 = \mathbf{e}_3 \times \mathbf{e}_1/\det\mathbf{J}, \mathbf{e}^3 = \mathbf{e}_1 \times \mathbf{e}_2/\det\mathbf{J},$$
$$\mathbf{e}_1 = \mathbf{e}^2 \times \mathbf{e}^3/\det\mathbf{J}, \mathbf{e}_2 = \mathbf{e}^3 \times \mathbf{e}^1/\det\mathbf{J}, \mathbf{e}_3 = \mathbf{e}^1 \times \mathbf{e}^2/\det\mathbf{J}, \tag{4.3}$$

where, by the properties of determinants,

$$\frac{1}{\det\mathbf{J}^{-1}} = \det\mathbf{J} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \mathbf{e}_3 \cdot \mathbf{e}_1 \times \mathbf{e}_2,$$
$$\det\mathbf{J}^{-1} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)} = \mathbf{e}^3 \cdot \mathbf{e}^1 \times \mathbf{e}^2. \tag{4.4}$$

By (4.3) and the properties of Jacobians (Margenau and Murphy 1956, pp. 19-20),

$$\mathbf{e}^1 = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}\frac{\partial\mathbf{x}}{\partial Y} \times \frac{\partial\mathbf{x}}{\partial Z} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}\left[\frac{\partial(y,z)}{\partial(Y,Z)}\mathbf{i} + \frac{\partial(z,x)}{\partial(Y,Z)}\mathbf{j} + \frac{\partial(x,y)}{\partial(Y,Z)}\mathbf{k}\right]$$
$$= \frac{\partial(X,Y,Z)}{\partial(x,y,z)}\left[\frac{\partial(x,y,z)}{\partial(x,Y,Z)}\mathbf{i} + \frac{\partial(y,z,x)}{\partial(y,Y,Z)}\mathbf{j} + \frac{\partial(z,x,y)}{\partial(z,Y,Z)}\mathbf{k}\right] = \frac{\partial X}{\partial x}\mathbf{i} + \frac{\partial X}{\partial y}\mathbf{j} + \frac{\partial X}{\partial z}\mathbf{k} = \nabla X,$$
$$\mathbf{e}^2 = \nabla Y, \mathbf{e}^3 = \nabla Z, \tag{4.5}$$

as obtained by DJ22. Thus, the contravariant basis vectors are the row vectors of $\mathbf{J}^{-1}$. Also, the $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$ vectors are normal vectors, i.e., normal to their X-, Y-, and Z-surfaces, respectively. Since $\mathbf{J}\,\mathbf{J}^{-1} = \mathbf{I}$, the 3x3 unit matrix,

$$\mathbf{e}_i \nabla X^j = \delta_i^j \tag{4.6}$$

where δ_i^j is the Kronecker delta. The Lagrangian continuity equation (3.16) implies that

$$\det\mathbf{J} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3 = \frac{\alpha}{\alpha_0} \tag{4.7}$$

(Lamb 1945, article 14; Salmon 1998, p. 6). The inverse of (4.7) is

$$\frac{\alpha_0}{\alpha} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)} = \nabla X \times \nabla Y \cdot \nabla Z = \mathbf{e}^1 \times \mathbf{e}^2 \cdot \mathbf{e}^3. \tag{4.8}$$

For later reference, we note that (4.3), (4.5) and (4.8) imply that

$$\mathbf{e}_1 = \frac{\alpha}{\alpha_0}\nabla Y \times \nabla Z, \mathbf{e}_2 = \frac{\alpha}{\alpha_0}\nabla Z \times \nabla X, \mathbf{e}_3 = \frac{\alpha}{\alpha_0}\nabla X \times \nabla Y. \tag{4.9}$$

A generic vector **A** can be expressed as

$$\begin{gathered}\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 \\ \text{or } \mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3\end{gathered} \tag{4.10}$$

where

$$A^i = \mathbf{A}\cdot\mathbf{e}^i \tag{4.11}$$

are the contravariant components, and

$$A_i = \mathbf{A}\cdot\mathbf{e}_i \tag{4.12}$$

are the covariant components of **A**. The scalar product of two vectors **A** and **B** is

$$\mathbf{A}\bullet\mathbf{B} = A^i\mathbf{e}_i\nabla X^j B_j = A^i B_j. \tag{4.13}$$

By the chain rule and (4.5), the gradient operator

$$\nabla = \nabla X\frac{\partial}{\partial X} + \nabla Y\frac{\partial}{\partial Y} + \nabla Z\frac{\partial}{\partial Z}. \tag{4.14}$$

The curl formula in non-orthogonal coordinates is obtained as follows (DJ22). From (4.5), (4.10), (4.8) and (4.9),

$$\begin{aligned}\nabla\times\mathbf{A} &\equiv \nabla\times\left(A_k\nabla X^k\right) = \frac{\partial A_k}{\partial X^j}\nabla X^j\times\nabla X^k \\ &= \frac{\alpha_0}{\alpha}\left(\frac{\partial A_3}{\partial Y} - \frac{\partial A_2}{\partial Z}\right)\mathbf{e}_1 + \frac{\alpha_0}{\alpha}\left(\frac{\partial A_1}{\partial Z} - \frac{\partial A_3}{\partial X}\right)\mathbf{e}_2 + \frac{\alpha_0}{\alpha}\left(\frac{\partial A_2}{\partial X} - \frac{\partial A_1}{\partial Y}\right)\mathbf{e}_3.\end{aligned} \tag{4.15}$$

Substituting $a\nabla\varphi$ for **A** in (4.15) yields

$$\frac{\alpha}{\alpha_0}\nabla a\times\nabla\varphi \equiv \frac{\partial(a,\varphi)}{\partial(Y,Z)}\mathbf{e}_1 + \frac{\partial(a,\varphi)}{\partial(Z,X)}\mathbf{e}_2 + \frac{\partial(a,\varphi)}{\partial(X,Y)}\mathbf{e}_3. \tag{4.16}$$

for the vector product of two gradients. By the scalar product of (4.16) with $\nabla\Theta$ and use of (4.13),

$$\frac{\alpha}{\alpha_0}\nabla\Theta\bullet\nabla a\times\nabla\varphi \equiv \frac{\partial(x,y,z)}{\partial(X,Y,Z)}\frac{\partial(\vartheta,a,\varphi)}{\partial(x,y,z)} = \frac{\partial(\vartheta,a,\varphi)}{\partial(X,Y,Z)} = \hat{\nabla}\Theta\bullet\hat{\nabla}a\times\hat{\nabla}\varphi. \tag{4.17}$$

where

$$\hat{\nabla} \equiv \mathbf{i}\frac{\partial}{\partial X} + \mathbf{j}\frac{\partial}{\partial Y} + \mathbf{k}\frac{\partial}{\partial Z}. \tag{4.18}$$

defines the gradient operator in *XYZ*-space. Eq. (4.17) provides the conversion of the scalar triple product of three gradients between *xyz*-space and *XYZ*-space.

5. Formulas for parcel velocity, vorticity and helicity

In this section we use Lagrangian coordinates to prove the velocity formula (3.5) for a parcel, P, that is in dry isentropic frictionless motion. Formulas for parcel vorticity and helicity follow from this expression.

The first step consists of pre-multiplying the frictionless equation of motion by $\mathbf{J}^{\mathrm{T}}$, the transpose of $\mathbf{J}$ (subscript T denotes transpose). This yields the Lagrangian equation of motion (Lamb 1945, article 13)

$$\begin{bmatrix} \partial x/\partial X & \partial y/\partial X & \partial z/\partial X \\ \partial x/\partial Y & \partial y/\partial Y & \partial z/\partial Y \\ \partial x/\partial Z & \partial y/\partial Z & \partial z/\partial Z \end{bmatrix} \begin{bmatrix} \partial u/\partial \tau \\ \partial v/\partial \tau \\ \partial w/\partial \tau \end{bmatrix}$$

$$= \begin{bmatrix} \partial x/\partial X & \partial y/\partial X & \partial z/\partial X \\ \partial x/\partial Y & \partial y/\partial Y & \partial z/\partial Y \\ \partial x/\partial Z & \partial y/\partial Z & \partial z/\partial Z \end{bmatrix} \begin{bmatrix} -\partial\sigma/\partial x + c_p\pi\partial\theta/\partial x \\ -\partial\sigma/\partial y + c_p\pi\partial\theta/\partial y \\ -\partial\sigma/\partial z + c_p\pi\partial\theta/\partial z \end{bmatrix}$$

$$= \begin{bmatrix} -\partial\sigma/\partial X + c_p\pi\partial\theta/\partial X \\ -\partial\sigma/\partial Y + c_p\pi\partial\theta/\partial Y \\ -\partial\sigma/\partial Z + c_p\pi\partial\theta/\partial Z \end{bmatrix} \tag{5.1}$$

by the chain rule. We see that $\hat{\nabla}x$, $\hat{\nabla}y$ and $\hat{\nabla}z$ are the column vectors of $\mathbf{J}^{\mathrm{T}}$. Hence, we can write (5.1) as

$$\hat{\nabla}x\frac{\partial u}{\partial\tau} + \hat{\nabla}y\frac{\partial v}{\partial\tau} + \hat{\nabla}z\frac{\partial w}{\partial\tau} = -\hat{\nabla}\sigma + \pi c_p\hat{\nabla}\theta. \tag{5.2}$$

Note that

$$\hat{\nabla}x\frac{\partial u}{\partial\tau} + \hat{\nabla}y\frac{\partial v}{\partial\tau} + \hat{\nabla}z\frac{\partial w}{\partial\tau} = \frac{\partial}{\partial\tau}\left(u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z\right) - u\hat{\nabla}\frac{\partial x}{\partial\tau} - v\hat{\nabla}\frac{\partial y}{\partial\tau} - w\hat{\nabla}\frac{\partial z}{\partial\tau}$$

$$= \frac{\partial}{\partial\tau}\left(u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z\right) - \hat{\nabla}\frac{q^2}{2} \tag{5.3}$$

since $\partial x/\partial\tau = u$, etc. Introducing (5.3) into (5.2) produces

$$\frac{\partial}{\partial\tau}\left(u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z\right) = \hat{\nabla}\left(\frac{q^2}{2} - \sigma\right) + \pi c_p\hat{\nabla}\theta. \tag{5.4}$$

In terms of the potential Φ and the cumulative (time-integrated) non-dimensional pressure Π, we can write (5.4) as

$$\frac{\partial}{\partial\tau}\left(u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z\right) = \frac{\partial}{\partial\tau}\left(\hat{\nabla}\Phi + \Pi c_p\hat{\nabla}\theta\right) \tag{5.5}$$

since $\hat{\nabla}\theta$ is a constant of the motion. By integrating (5.5) from the initial time τ_0 to the current time τ and applying initial conditions, we obtain

$$u\hat{\nabla}x + v\hat{\nabla}y + w\hat{\nabla}z - \mathbf{v}_0 = \hat{\nabla}\Phi + \Pi c_p\hat{\nabla}\theta \tag{5.6}$$

where

$$\Phi(\tau) = \int_{\tau'=\tau_0}^{\tau}\left[\frac{q^2(\tau')}{2} - \sigma(\tau')\right]d\tau' \tag{5.7}$$

and

$$\Pi(\tau) = \int_{\tau_0}^{\tau}\pi(\tau')d\tau' \tag{5.8}$$

in terms of Lagrangian integrals (following a parcel). This is Weber's transformation (Lamb 1945, article 15) as generalized by Serrin (1959), Dutton (1976, p. 388) and Davies-Jones (2015b).

To obtain an explicit formula for a parcel's current velocity, we follow Dutton (1976, p. 388) by pre-multiplying (5.6) by $(\mathbf{J}^{\mathrm{T}})^{-1}$. The column vectors of $(\mathbf{J}^{\mathrm{T}})^{-1}$ are ∇X, ∇Y and ∇Z. The following identities apply. For any differentiable scalars a and φ,

$$\left(\mathbf{J}^{\mathrm{T}}\right)^{-1}a\hat{\nabla}\varphi = \begin{bmatrix} \partial X/\partial x & \partial Y/\partial x & \partial Z/\partial x \\ \partial X/\partial y & \partial Y/\partial y & \partial Z/\partial y \\ \partial X/\partial z & \partial Y/\partial z & \partial Z/\partial z \end{bmatrix}\begin{bmatrix} a\,\partial\varphi/\partial X \\ a\,\partial\varphi/\partial Y \\ a\,\partial\varphi/\partial Z \end{bmatrix}$$

$$= \begin{bmatrix} a\,\partial\varphi/\partial x \\ a\,\partial\varphi/\partial y \\ a\,\partial\varphi/\partial z \end{bmatrix} = a\nabla\varphi = a\frac{\partial\varphi}{\partial X}\nabla X + a\frac{\partial\varphi}{\partial Y}\nabla Y + a\frac{\partial\varphi}{\partial Z}\nabla Z \tag{5.9}$$

by the chain rule and (4.14). For any vector $\mathbf{A} \equiv A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$,

$$\left(\mathbf{J}^{\mathrm{T}}\right)^{-1}\mathbf{A} = A\nabla X + B\nabla \mathrm{Y} + C\nabla Z \tag{5.10}$$

by the chain rule. Thus, the result of premultiplying (5.6) by $(\mathbf{J}^{\mathrm{T}})^{-1}$ is

$$\mathbf{v}(\tau) = \left(u_0 + \frac{\partial\Phi}{\partial X} + \Pi c_p\frac{\partial\theta}{\partial X}\right)\nabla \mathrm{X} + \left(v_0 + \frac{\partial\Phi}{\partial Y} + \Pi c_p\frac{\partial\theta}{\partial Y}\right)\nabla \mathrm{Y}$$

$$+\left(w_0 + \frac{\partial\Phi}{\partial Z} + \Pi c_p \frac{\partial\theta}{\partial Z}\right)\nabla Z \tag{5.11}$$

after using (4.14). The coefficients of ∇X, ∇Y and ∇Z are the covariant wind components. The formula for **v** is partly implicit because the potential Φ contains the windspeed squared in its integrand in (5.7). Since (5.11) is (3.5) and (3.11) expanded via the chain rule, we have proven (3.5) and (3.11). Note that in (5.11) the terms involving Φ and θ comprise $\mathbf{v}_\Phi$ and $\mathbf{v}_{BC}$, respectively, and the remaining terms compose $\mathbf{v}_{BT}$.

Applying (4.15) to the wind formula (5.11) gives us the **w**-formula,

$$\mathbf{w}(\tau) = \left[\frac{\partial w_0}{\partial Y} - \frac{\partial v_0}{\partial Z} + \frac{\partial(\Pi, c_p\theta)}{\partial(Y,Z)}\right]\mathbf{e}_1 + \left[\frac{\partial u_0}{\partial Z} - \frac{\partial w_0}{\partial X} + \frac{\partial(\Pi, c_p\theta)}{\partial(Z,X)}\right]\mathbf{e}_2$$

$$+\left[\frac{\partial v_0}{\partial X} - \frac{\partial u_0}{\partial Y} + \frac{\partial(\Pi, c_p\theta)}{\partial(X,Y)}\right]\mathbf{e}_3, \tag{5.12}$$

where the coefficients of the $\mathbf{e}_i$ are the contravariant components of **w**. Using (4.16) in (3.12) obtains the same result. The terms involving θ constitute the baroclinic vorticity and the remaining terms comprise the baroclinic vorticity.

The scalar product of (5.11) and (5.12) yields the parcel-helicity formula,

$$h(\tau) = \left[\frac{\partial w_0}{\partial Y} - \frac{\partial v_0}{\partial Z} + \frac{\partial(\Pi, c_p\theta)}{\partial(Y,Z)}\right]\mathbf{e}_1 \bullet \nabla X\left(u_0 + \frac{\partial\Phi}{\partial X} + \Pi c_p \frac{\partial\theta}{\partial X}\right)$$

$$+\left[\frac{\partial u_0}{\partial Z} - \frac{\partial w_0}{\partial X} + \frac{\partial(\Pi, c_p\theta)}{\partial(Z,X)}\right]\mathbf{e}_2 \bullet \nabla Y\left(v_0 + \frac{\partial\Phi}{\partial Y} + \Pi c_p \frac{\partial\theta}{\partial Y}\right)$$

$$+\left[\frac{\partial v_0}{\partial X} - \frac{\partial u_0}{\partial Y} + \frac{\partial(\Pi, c_p\theta)}{\partial(X,Y)}\right]\mathbf{e}_3 \bullet \nabla Z\left(w_0 + \frac{\partial\Phi}{\partial Z} + \Pi c_p \frac{\partial\theta}{\partial Z}\right), \tag{5.13}$$

where $\mathbf{e}_1 \bullet \nabla X = \mathbf{e}_2 \bullet \nabla Y = \mathbf{e}_3 \bullet \nabla Z = 1$ by (4.6). This is the integral of (3.2) for isentropic frictionless motion. Note that (5.13) is simply an expanded form of (3.13). More succinctly,

$$h(\tau) = (\alpha/\alpha_0)\left(u_0\nabla X + v_0\nabla Y + w_0\nabla Z + \nabla\Phi + \Pi c_p\nabla\theta\right) \bullet \left(\nabla u_0 \times \nabla X + \nabla v_0 \times \nabla Y + \nabla w_0 \times \nabla Z + \nabla\Pi \times c_p\nabla\theta\right) \tag{5.14}$$

where the second and third bracketed expressions on the right side are $\mathbf{v}(\tau)$ and $\boldsymbol{\omega}(\tau)$, respectively.

We now verify (5.14) by recovering (3.2) when $\mathbf{G} \equiv \mathbf{0}$. Using the triple-product identity (4.17) yields

$$h(\tau) = \left(u_0\hat{\nabla}X + v_0\hat{\nabla}Y + w_0\hat{\nabla}Z + \hat{\nabla}\Phi + \Pi c_p\hat{\nabla}\theta\right) \bullet$$
$$\left(\hat{\nabla}u_0 \times \hat{\nabla}X + \hat{\nabla}v_0 \times \hat{\nabla}Y + \hat{\nabla}w_0 \times \hat{\nabla}Z + \hat{\nabla}\Pi \times c_p\hat{\nabla}\theta\right). \tag{5.15}$$

Differentiating (5.15) gives

$$\frac{\partial h}{\partial \tau} = \left(\hat{\nabla}u_0 \times \hat{\nabla}X + \hat{\nabla}v_0 \times \hat{\nabla}Y + \hat{\nabla}w_0 \times \hat{\nabla}Z + \hat{\nabla}\Pi \times c_p\hat{\nabla}\theta\right) \bullet \left(\hat{\nabla}\frac{\partial \Phi}{\partial \tau} + \pi c_p\hat{\nabla}\theta\right)$$
$$+\hat{\nabla}\pi \times c_p\hat{\nabla}\theta \bullet \left(u_0\hat{\nabla}X + v_0\hat{\nabla}Y + w_0\hat{\nabla}Z + \hat{\nabla}\Phi + \Pi c_p\hat{\nabla}\theta\right). \tag{5.16}$$

Using (4.17) again provides

$$\frac{\partial h}{\partial \tau} = \mathbf{w} \bullet \nabla\frac{\partial \Phi}{\partial \tau} + \mathbf{w} \bullet \pi c_p\nabla\theta + \frac{\alpha}{\alpha_0}\mathbf{v} \bullet \nabla\pi \times c_p\nabla\theta, \tag{5.17}$$

which is (3.2) with $\mathbf{G} \equiv \mathbf{0}$.

6. Isentropic motion in a horizontally uniform, constant environment

We can simplify the formulas for unsteady isentropic motion if we stipulate a horizontally uniform, constant environment at upstream infinity so that $w_0 = 0$ and u_0, v_0, and other environmental variables are functions of just Z. We assume that all the parcels currently in the storm were initially in the environment so subscript 0 now indicates environmental value as well as initial value. Initially the covariant and contravariant bases are orthonormal and coincident. Since no heat is transferred to or from a parcel, θ is a function of Z alone. The PV is zero initially and remains zero (see appendix A). In this case, (3.5) and (3.11) reduce to

$$\mathbf{v} = u_0\nabla X + v_0\nabla Y + \nabla\Phi + \Pi c_p\nabla\theta. \tag{6.1}$$

Physical interpretation of (6.1) is complicated by the fact that the contravariant basis vectors, ∇X and ∇Y are not generally tangential to the Z-surface (∇Z is of course always normal to this surface).

We now twist the Lagrangian coordinate system so that in each Z-surface the s_0-axis is in the storm-relative environmental wind direction $\beta_0(Z)$ and the n_0-axis is 90° to the left of it (DJ17, DJ22). The Z-axis is unchanged. In the rotated Lagrangian system, n_0 labels a streamline within a Z-surface, and s_0 labels a parcel along its streamline. The transformation from the (X, Y, Z) coordinates to the turned ones (s_0, n_0, Z) is

$$\begin{bmatrix} s_0 \\ n_0 \\ Z \end{bmatrix} = \begin{bmatrix} \cos\beta_0 & \sin\beta_0 & 0 \\ -\sin\beta_0 & \cos\beta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{6.2}$$

where the square matrix is a rotation matrix **R**. [In DJ22, s_0 and n_0 were named $\bar{X}$ and $\bar{Y}$.] The inverse of **R** is its transpose, and its determinant is 1. The storm-relative environmental wind components in the original and twisted Lagrangian systems are related by

$$\begin{bmatrix} u_0 \\ v_0 \\ 0 \end{bmatrix} = q_0 \begin{bmatrix} \cos\beta_0 \\ \sin\beta_0 \\ 0 \end{bmatrix} \tag{6.3}$$

where $q_0(Z)$ is the storm-relative environmental windspeed. By the chain rule,

$$\nabla = \nabla s_0 \frac{\partial}{\partial s_0} + \nabla n_0 \frac{\partial}{\partial n_0} + \nabla Z \frac{\partial}{\partial Z} \tag{6.4}$$

and by (6.2),

$$\begin{aligned} \nabla X &= \nabla s_0 \cos\beta_0 - \nabla n_0 \sin\beta_0 - (s_0 \sin\beta_0 \cos\beta_0 + n_0 \cos^2\beta_0) q_0 (d\beta_0/dZ) \nabla Z, \\ \nabla Y &= \nabla s_0 \sin\beta_0 + \nabla n_0 \cos\beta_0 + (s_0 \sin\beta_0 \cos\beta_0 - n_0 \sin^2\beta_0) q_0 (d\beta_0/dZ) \nabla Z. \end{aligned} \tag{6.5}$$

Hence from (6.1), (6.3), and (6.5), the velocity formula becomes

$$\begin{aligned} \mathbf{v} &= q_0 \nabla s_0 + \nabla\Phi - n_0 q_0 \nabla\beta_0 + \Pi c_p \nabla\theta \\ &= \nabla\phi + \chi \nabla Z \end{aligned} \tag{6.6}$$

where

$$\phi \equiv q_0 s_0 + \Phi, \tag{6.7}$$

$$\chi \equiv -\frac{dq_0}{dZ} s_0 - q_0 \frac{d\beta_0}{dZ} n_0 + c_p \frac{d\theta}{dZ} \Pi. \tag{6.8}$$

Note that $-q_0 d\beta_0/dZ$ and dq_0/dZ are, respectively, the environmental streamwise and crosswise vorticity (DJ84). Eq. (6.8) is equivalent to eq. 76 in DJ22. From the curl of (6.6),

$$\boldsymbol{\omega} = \nabla\chi \times \nabla Z \tag{6.9}$$

so the vortex lines lie in the Z-surfaces and the contours of constant χ within a Z-surface are the vortex lines. The addition of $q_0 s_0$ to Φ is equivalent to superimposing the potential of the environmental wind on the potential flow $\nabla\Phi$. For example, $\nabla\phi$ could represent a combination of uniform flow and a potential vortex. From (6.6) and (6.9), the parcel helicity

$$h \equiv \mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \nabla\phi = \frac{\alpha}{\alpha_0} \nabla\chi \times \nabla Z \bullet \nabla\phi. \tag{6.10}$$

We now verify that (6.10) is the integral of (3.2) in the special case of isentropic dragless motion in a horizontally uniform environment. Note that

$$\frac{D\chi}{Dt} = \pi c_p \frac{d\theta}{dZ} \tag{6.11}$$

from (6.8) and

$$\frac{D\phi}{Dt} = \frac{D(q_0 s_0 + \Phi)}{Dt} = q^2/2 - \sigma \tag{6.12}$$

via (3.3). Taking the material derivative of (6.10) and introducing (2.12), (A2), (6.12) and (6.6) gives

$$\frac{Dh}{Dt} = \nabla\phi \bullet \frac{D\mathbf{w}}{Dt} + \mathbf{w} \bullet \frac{D}{Dt}\nabla\phi$$

$$= \nabla\phi \bullet (\mathbf{w} \cdot \nabla)\mathbf{v} + \frac{\alpha}{\alpha_0}\nabla\phi \bullet \nabla\pi \times c_p \frac{d\theta}{dZ}\nabla Z + \mathbf{w} \bullet \nabla\frac{D\phi}{Dt} - \nabla\phi \bullet (\mathbf{w} \cdot \nabla)\mathbf{v}$$

$$= \mathbf{w} \bullet \nabla\left(\frac{q^2}{2} - \sigma\right) + \frac{\alpha}{\alpha_0}\mathbf{v} \bullet \nabla\pi \times \nabla(c_p\theta), \tag{6.13}$$

which is (3.2).

7. Steady isentropic flow with a horizontally uniform environment

In this section, we use natural coordinates to show that the "river-bend effect" (Shapiro 1972; Adlerman et al. 1999; Davies-Jones et al. 2001; DJ17, DJ22) is hidden in the first term on the right of (3.2). To reveal this effect and to recover Scorer's (1997) streamwise-vorticity equation, we assume the same flow is steady with a horizontally uniform environment. Because a steady, horizontally uniform environment is impossible with a friction force (Davies-Jones 2021), we again omit **F**. Since PV is conserved and is zero in the environment, PV is zero everywhere. Following Kuo (1966), we stipulate unstable stratification at low levels ($d\theta/dZ < 0$). In our analysis, this is necessary for parcels with overall upward (downward) displacements to be warmer (cooler) than the environment at the same height.

Steady processes should become relevant as thunderstorms mature into quasi-steady supercells. Existence of a steady solution is important because it represents an equilibrium point in a phase space. This fixed point may be associated with an attractor, a set of nearby points towards which the dynamical system evolves eventually, provided that the initial state lies in the attractor's basin of attraction. Thus, an unsteady solution could characterize a cyclic supercell if it ends up orbiting the equilibrium point.

Assuming steady isentropic flow with a horizontally uniform environment leads to major simplifications. As in the unsteady case (section 6), zero PV stipulates that the vortex lines lie in the isentropic surfaces. The trajectories are now streamlines that lie in the isentropic surfaces because $\mathbf{v}\bullet\nabla\theta = 0$. Hence, the osculating planes are tangent to the *Z*-surfaces.

We begin by deriving expressions for streamwise vorticity and abnormality in natural coordinates (DJ17). At any instant, let **t** be the unit vector in the parcel's instantaneous direction of travel, **n** be the unit normal in its osculating plane (the plane of adjacent tangents to its streamline), and **b** be the unit binormal such that **t**, **n** and **b** are an orthonormal set of vectors (Scorer 1997, p. 75; Haltiner and Martin, 1957, p. 170). Let *ds*, *dn*, *db* denote infinitesimal distances from the parcel in these three directions. In terms of *Z*, $\mathbf{b} = \nabla Z/|\nabla Z|$ and $\partial/\partial b = (dZ/db)\,\partial/\partial Z$. In natural coordinates, the velocity is

$$\mathbf{v} = q\mathbf{t}, \tag{7.1}$$

the parcel streamwise vorticity is $\boldsymbol{\omega}\bullet\mathbf{t}$, and the generalized abnormality is

$$\lambda \equiv \frac{h}{q^2} = \frac{\boldsymbol{\omega}\cdot\mathbf{t}}{q}. \tag{7.2}$$

Furthermore, the gradient operator is

$$\nabla = \mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n} + \mathbf{b}\frac{\partial}{\partial b} \tag{7.3}$$

The vorticity is

$$\boldsymbol{\omega} \equiv \nabla\times\mathbf{v} = \left(\mathbf{t}\frac{\partial}{\partial s} + \mathbf{n}\frac{\partial}{\partial n} + \mathbf{b}\frac{\partial}{\partial b}\right)\times q\mathbf{t}$$

$$= \mathbf{t}\times\frac{\partial(q\mathbf{t})}{\partial s} + \mathbf{n}\times\frac{\partial(q\mathbf{t})}{\partial n} + \mathbf{b}\times\frac{\partial(q\mathbf{t})}{\partial b}$$

$$= q\left(\mathbf{t}\times\frac{\partial\mathbf{t}}{\partial s} + \mathbf{n}\times\frac{\partial\mathbf{t}}{\partial n} + \mathbf{b}\times\frac{\partial\mathbf{t}}{\partial b}\right) - \frac{\partial q}{\partial n}\mathbf{b} + \frac{\partial q}{\partial b}\mathbf{n}. \tag{7.4}$$

(DJ17). By one of the Frénet equations,

$$\frac{\partial\mathbf{t}}{\partial s} = \kappa\mathbf{n} \tag{7.5}$$

where κ is the streamline curvature, which is positive for left-turning motion and negative for right-turning motion. By differentiation of the identity $\mathbf{t}\cdot\mathbf{t} = 1$, we find that

$$\mathbf{t}\cdot\frac{\partial\mathbf{t}}{\partial s}=\mathbf{t}\cdot\frac{\partial\mathbf{t}}{\partial n}=\mathbf{t}\cdot\frac{\partial\mathbf{t}}{\partial b}=\ 0. \tag{7.6}$$

Via these identities, some vector identities, and (7.5),

$$\mathbf{t}\times\frac{\partial\mathbf{t}}{\partial s}=\mathbf{t}\times\kappa\mathbf{n}=\kappa\mathbf{b},$$

$$\mathbf{n}\times\frac{\partial\mathbf{t}}{\partial n}=(\mathbf{b}\times\mathbf{t})\times\frac{\partial\mathbf{t}}{\partial n}=\left(\frac{\partial\mathbf{t}}{\partial n}\cdot\mathbf{b}\right)\mathbf{t}-\left(\frac{\partial\mathbf{t}}{\partial n}\cdot\mathbf{t}\right)\mathbf{b}=\left(\frac{\partial\mathbf{t}}{\partial n}\cdot\mathbf{b}\right)\mathbf{t},$$

$$\mathbf{b}\times\frac{\partial\mathbf{t}}{\partial b}=(\mathbf{t}\times\mathbf{n})\times\frac{\partial\mathbf{t}}{\partial b}=\left(\frac{\partial\mathbf{t}}{\partial b}\cdot\mathbf{t}\right)\mathbf{n}-\left(\frac{\partial\mathbf{t}}{\partial b}\cdot\mathbf{n}\right)\mathbf{t}=-\left(\frac{\partial\mathbf{t}}{\partial b}\cdot\mathbf{n}\right)\mathbf{t}. \tag{7.7}$$

Introducing (7.7) into (7.4) yields

$$\boldsymbol{\omega}=q\left(\frac{\partial\mathbf{t}}{\partial n}\cdot\mathbf{b}-\frac{\partial\mathbf{t}}{\partial b}\cdot\mathbf{n}\right)\mathbf{t}+\frac{\partial q}{\partial b}\mathbf{n}+\left(\kappa q-\frac{\partial q}{\partial n}\right)\mathbf{b} \tag{7.8}$$

(DJ17). The binormal vorticity

$$\boldsymbol{\omega}\cdot\mathbf{b}=\kappa q-\frac{\partial q}{\partial n}. \tag{7.9}$$

It consists of 'curvature vorticity' κq and 'shear vorticity' $-\partial q/\partial n$. By the stipulation of zero PV, the binormal vorticity is zero. Thus,

$$\partial q/\partial n=\kappa q. \tag{7.10}$$

The equation of motion for this flow is

$$\nabla B+\boldsymbol{\omega}\times\mathbf{v}=\pi c_p\nabla\theta. \tag{7.11}$$

from (2.11) and (2.8). By introducing (7.1), (7.3) and (7.5) into (7.11), we obtain the equations of motion in the tangential, normal, and binormal directions. These are

$$\frac{\partial B}{\partial s}=0,$$

$$\frac{\partial B}{\partial n}=\kappa q^2+\frac{\partial\sigma}{\partial n}=0,$$

$$0=-\frac{\partial\sigma}{\partial b}+\pi c_p\frac{\partial\theta}{\partial b}. \tag{7.12}$$

They state that in steady flow (i) in each Z-surface, the Bernoulli function, B, is constant, (ii) the normal pressure-gradient force balances the centrifugal force, (iii) there is a force balance in the binormal direction that could explain why a long-lived tower of a supercell sometimes has a surprisingly laminar and striated appearance (e.g., Davies-Jones et al. 1976; Markowski and Richardson 2010, p. 217). [The look of the cloud face suggests helical motion along axes parallel to the striations.]. The total energy, B, of a parcel, the sum of its specific kinetic energy, $q^2/2$, its specific enthalpy, c_pT, and its specific potential energy, gz, is conserved, Thus, a parcel in steady isentropic motion without drag can acquire large kinetic energy (KE) only through a compensating decrease in its static energy (Davies-Jones 2015a).

We can now show that the helicity equation (3.2) for this case is Scorer's equation. Via (7.3), (7.8) and (7.10), the first term on the right of (3.2) becomes

$$\mathbf{w}\cdot\nabla\left(\frac{q^2}{2}-\boldsymbol{\sigma}\right)=\left(\mathbf{w}\cdot\mathbf{t}\frac{\partial}{\partial s}+\mathbf{w}\cdot\mathbf{n}\frac{\partial}{\partial n}\right)\left(\frac{q^2}{2}-\boldsymbol{\sigma}\right)$$

$$=\mathbf{w}\cdot\mathbf{t}\frac{\partial}{\partial s}\left(\frac{q^2}{2}-\boldsymbol{\sigma}\right)+\mathbf{w}\cdot\mathbf{n}\left(\kappa q^2-\frac{\partial\boldsymbol{\sigma}}{\partial n}\right). \tag{7.13}$$

Inserting (7.13) into (3.2) produces

$$\frac{Dh}{Dt}=\frac{\partial}{\partial s}\left(\frac{q^2}{2}-\sigma\right)\mathbf{w}\cdot\mathbf{t}+\left(\kappa q^2-\frac{\partial\sigma}{\partial n}\right)\mathbf{w}\cdot\mathbf{n}+\frac{\alpha}{\alpha_0}\mathbf{v}\cdot\nabla\pi\times c_p\nabla\theta. \tag{7.14}$$

The first term on the right of (7.14) represents the parcel's rate of change of helicity due to increase of kinetic energy minus static energy along a streamline times streamwise vorticity. The second term describes the helicity rate of change due to the turning streamwise of normal vorticity in curved flow (the river-bend effect). The pressure is lower on the inside of the bend so $-\partial\sigma/\partial n$ has the same sign as κq^2. In a left-hand bend positive transverse vorticity is turned streamwise. The third term is the scalar product of wind with the baroclinic **w**.

From (2.8) and the constancy of B on a Z-surface,

$$\frac{q^2}{2}-\sigma=q^2-B(Z). \tag{7.15}$$

Hence,

$$\frac{\partial}{\partial s}\left(\frac{q^2}{2}-\sigma\right)=\frac{\partial q^2}{\partial s},\frac{\partial}{\partial n}\left(\frac{q^2}{2}-\sigma\right)=2\kappa q^2 \tag{7.16}$$

by (7.10). After inserting (7.16) and (7.2) into (7.14), we get

$$\frac{Dh}{Dt}=q\frac{\partial}{\partial s}(\lambda q^2)=\lambda q\frac{\partial q^2}{\partial s}+2\kappa q^2\mathbf{w}\cdot\mathbf{n}+\frac{\alpha}{\alpha_0}\mathbf{v}\cdot\nabla\pi\times c_p\nabla\theta. \tag{7.17}$$

After dividing by q^2 and simplifying, we obtain

$$q\frac{\partial\lambda}{\partial s}=2\kappa\mathbf{w}\cdot\mathbf{n}+\frac{\alpha}{\alpha_0}\frac{\mathbf{v}}{q^2}\cdot\nabla\pi\times c_p\nabla\theta. \tag{7.18}$$

This equation for λ is the same as Scorer's (1997, p. 78) 'streamwise-vorticity equation' [since $\lambda = \omega_s/q$, $\mathbf{R}\times(\mathbf{g} - \mathbf{f})$ in Scorer's notation is $c_p\nabla\pi\times\nabla\theta$ in ours and $\alpha/\alpha_0 = 1$ in Scorer's Boussinesq flow]. Scorer discusses the physical interpretation of this equation in detail. For a steady BF, (7.18) correctly predicts that λ = const. In steady isentropic flow, λ changes along a streamline due to the river-bend effect acting on barotropic and baroclinic transverse vorticity and due to direct baroclinic generation of λ.

The integral of (7.18) is

$$\lambda = -\frac{d\beta_0}{dZ} + \int_{-\infty}^{s} \frac{2\kappa\mathbf{w}\cdot\mathbf{n}}{q} ds' + \int_{-\infty}^{s} \frac{\alpha\mathbf{t}\cdot\nabla\pi\times c_p\nabla\theta}{\alpha_0 q^2} ds' \tag{7.19}$$

where the integrands are evaluated at s'. The first term on the right of (7.19) is the initial abnormality. In a BF, it is the only nonzero term. The second and third terms are path dependent. They are the abnormalities associated with curved flow and with baroclinic generation of vorticity. The overall increase in λ depends on the parcel's dwell time in curved flow and in temperature gradients within its Z-surface.

The expression for parcel helicity is

$$h(s) = q^2(s)\left[\left(-\frac{d\beta_0}{dZ}\right) + \int_{-\infty}^{s} \frac{2\kappa\mathbf{w}\cdot\mathbf{n}}{q} ds' + \int_{-\infty}^{s} \frac{\alpha\mathbf{t}\cdot\nabla\pi\times c_p\nabla\theta}{\alpha_0 q^2} ds'\right]. \tag{7.20}$$

For scale analysis of (7.20) applied to a parcel in a tornado, we assume $q_0 \sim 10$ m s^{-1}, $d\beta_0/dZ \sim 10^{-3}$ rad m^{-1}, $dq_0/dZ \sim 10^{-2}$ s^{-1}, $q \sim 100$ m s^{-1} and tornado vorticity and helicity magnitudes of 1 s^{-1} and 100 m s^{-2}. The environmental parcel helicity, $-q_0^2 d\beta_0/dZ$, is about 0.1 m s^{-2}. The first term on the right of (7.20) is equal to the environmental parcel helicity times an amplification factor q^2/q_0^2 of roughly 100, which accounts for speed increase and for streamwise vortex stretching. It is only around 10 m s^{-2}, which is an order of magnitude smaller than the parcel helicity in a strong tornado. Since the results of Davies-Jones (2025b) indicate that the frictional terms in the helicity fail to explain the tornado's tremendous helicity, the last two terms in (7.20) must account for almost all of the tornado helicity.

There is one loose end that needs to be tidied up. Since the flow is steady, the initial time τ_0 for computing the integrals that define the quantities Φ and Π is infinitely long ago. This is problematic because the integrands are nonzero in the upstream environment. We circumvent this dilemma as follows. After substituting for $q^2/2 - \sigma$ from (7.15), (5.7) becomes

$$\Phi(\tau) = \int_{\tau'=\tau_0}^{\tau} [q^2(\tau') - B(Z)]d\tau' . \tag{7.21}$$

For steady motion, let us redefine Φ and Π as

$$\Phi(\tau) = \int_{\tau'=\tau_0}^{\tau} \left[q^2(\tau') - q_0^2(Z)\right]d\tau' , \tag{7.22}$$

and

$$\Pi(\tau) = \int_{\tau'=\tau_0}^{\tau} [\pi(\tau') - \pi_0(Z)]d\tau' . \tag{7.23}$$

Since the integrands in (7.22) and (7.23) are zero far upstream, Φ and Π now become independent of τ_0 as $\tau_0 \to -\infty$. Because we have added functions of the form $F(Z, \tau)$ to Φ and Π, the velocity formula (6.6) is now wrong. However, the vorticity formula (6.9) and the helicity formula (6.10) are still correct because they involve Φ and Π only in the forms $\partial\Phi/\mathrm{d}s_0$, $\partial\Phi/\mathrm{d}n_0$, $\partial\Pi/\mathrm{d}s_0$ and $\partial\Pi/\mathrm{d}n_0$ and these quantities are unaffected by the changes.

9. Concluding remarks

Previous attempts to explain tornadogenesis have been based on analyzing the tornado's large vorticity or circulation. The role of irrotational flow is invisible in this vorticity perspective. The present study investigates tornado formation from a helicity perspective, which fully incorporates the irrotational part of the wind. Neither perspective deals with the tricky problem of how parcel trajectories are altered by developing rotation and vice versa.

In the special case of dry frictionless isentropic motion, the covariant wind components are obtained herein. These are key to obtaining formulas for the parcel helicity h, which is the scalar product of **w** (dilatation times vorticity) and storm-relative velocity, q, and a new quantity λ, the generalized abnormality, which is h/q^2. Important Lagrangian integrals (following a parcel) that appear in these formulas are Π, a parcel's accumulated nondimensional pressure and Φ, the integral of a parcel's kinetic energy minus its enthalpy and its potential energy. For steady flow, the differential equation governing λ is Scorer's

(1997) streamwise-vorticity equation. In this special case, the formula for λ is the integral of Scorer's equation.

In a Beltrami flow, the abnormality is constant. Stretching of streamwise vorticity along the streamlines and wind intensification increases parcel helicity by the ratio of the parcel's current kinetic energy to its environmental kinetic energy. This amplification of environmental parcel helicity is sufficient to explain rotation of a supercell's updraft.

However, for a parcel entering a strong tornado, λ has increased tenfold over its environmental value so its helicity is far greater than can be explained by Beltrami flow. Thus, we must consider mechanisms that generate λ. Section 7 finds that, in steady isentropic flow, λ increases along a streamline due to the river-bend effect acting on positive barotropic and baroclinic transverse vorticity and due to direct baroclinic generation of λ.

The storm-relative parcel velocity, **v,** plays a fundamental role in the theory presented herein. In dry isentropic inviscid flow, **v** decomposes according to (3.11) and (3.5) into potential, barotropic, and baroclinic parts (denoted by subscripts Φ, BT, and BC, respectively). The velocity parts are $\mathbf{v}_\Phi = \nabla\Phi$, $\mathbf{v}_{BT} = \mathbf{v}_0 \bullet (\nabla\mathbf{X})$ where $\mathbf{v}_0$ and $\mathbf{X}$ are the parcel's initial velocity and position vector, and $\mathbf{v}_{BC} = \Pi c_p \nabla\theta$, where θ is its potential temperature. The barotropic and baroclinic vorticities are the curls of $\mathbf{v}_{BT}$ and $\mathbf{v}_{BC}$. Parcel helicity decomposes into six scalar products [see (3.21)] since there are three partial velocities and two partial **w**-vectors. The product $\mathbf{w}_{BT} \bullet \mathbf{v}_{BT}$ is simply the initial helicity. The velocity induced by baroclinic vorticity is perpendicular to baroclinic vorticity, hence $\mathbf{w}_{BC} \bullet \mathbf{v}_{BC}=0$. This is expected since wind and vorticity are mutually perpendicular in a simple model of a density current. In an idealized model of a supercell in a horizontally uniform environment, the flow conserves zero potential vorticity ($\mathbf{w} \bullet \nabla\theta = 0$). Then $\mathbf{w}_{BT} \bullet \mathbf{v}_{BC}=0$ and the gain in parcel helicity is given by $h-h_0=\mathbf{w}_{BT} \bullet \nabla\Phi+\mathbf{w}_{BC} \bullet \mathbf{v}$.

In the simplest case when the flow is frictionless and homentropic (constant θ), the current helicity of a parcel exceeds its initial helicity by the scalar product of its current (barotropic) vorticity with $\nabla\Phi$. This term is sufficient to explain the helicity of barotropic vortices. Since $\partial\Phi/\partial\tau$ gets very large in and near a tornado, the main $\mathbf{w}_{BT} \bullet \nabla\Phi$ contribution to helicity may occur in close proximity to the tornado and may be poorly sampled by diagnostic investigations of simulated supercells.

When the flow is frictionless and isentropic and the environment is horizontally uniform, we can superimpose the potential of the environmental wind on Φ. Then $h = \mathbf{w} \bullet \nabla\phi$ where ϕ is the combined potential. Thus, only the irrotational wind $\mathbf{v}_\phi \equiv \nabla\phi$ contributes to parcel helicity.

The vorticities of parcels flowing at low heights into a mesocyclone have been found previously to originate from predominantly baroclinic (i.e., storm-generated) vorticity rather than from stretching and twisting of environmental vorticity (e.g., Dahl 2015). Baroclinic generation of vorticity increases with sharper gradients of cumulative temperature on a material Z-surface and stronger unstable environmental stratification. If baroclinic vorticity is dominant, then by the above formula parcel helicity is equal to the scalar product of baroclinic $\mathbf{w}$ with $\mathbf{v}_\phi$, which can approximate potential flow around a rotating updraft in a helical environment.

In the MR14 idealized simulation, a parcel's baroclinic vorticity is generated primarily in the transverse direction and turned streamwise during left-turning subsidence in a rear-flank downdraft (RFD) that wraps cyclonically around the mesolow. Atmospheric observations indicate that the deepest descents to near ground occur from 1-2 km AGL in this region (Bartos et al., 2022). Subsidence packs the Z-surfaces closer together. Combination of this effect and streamline confluence increases near-ground wind speed according to mass conservation. As a parcel exits the RFD along the ground and spirals into a tornado, its helicity increases greatly due to acceleration and to streamwise stretching of its streamwise vorticity.

In steady inviscid isentropic flow, the Bernoulli function (sum of specific KE and static energy) is constant on each isentropic surface, so amplification of a parcel's specific KE is accompanied by a decline in its static energy. Imagine a parcel that descends from 500 m AGL to ground into a tornado. If the process is isothermal as well as isentropic, it loses $5\text{x}10^3$ m^2 s^{-2} in potential energy but no enthalpy. If energy is conserved, its kinetic energy gain is sufficient for winds of 100 m s^{-1}. In this hypothetical example, the parcel's pressure at the ground in the tornado would be the same as the ambient pressure at 500 m AGL. If the ambient cloud base were at 500 m, the condensation funnel would reach the ground. Within the tornado itself, there can be significant loss of Bernoulli function aloft at a vortex breakdown due to nonlocal forcing provided principally by divergence of the Lamb vector [see (2.13)].

The friction term in the helicity budget acts to decrease the helicity of near-ground parcels (Davies-Jones 2025b), so the friction term in the helicity equation cannot account for large parcel helicities in tornadoes. Nevertheless, it plays a major role in tornadogenesis by altering near-ground trajectories. Frictional interaction of the flow with the ground is needed for the final intensification of a tornado-scale vortex into a tornado that far exceeds the thermodynamic speed limit (Fiedler 1994). Surface friction increases convergence into a vortex, thereby enabling parcels to come closer to the rotation axis before rising and hence to spin faster because of partial conservation of angular momentum (Fiedler and Rotunno 1986).

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APPENDIX A

Potential-vorticity equation

Here we derive an important identity and use it to show that the potential vorticity $\mathbf{w}\bullet\nabla\Theta$ of a parcel in frictionless motion is conserved for any constant of the motion Θ that is a function of only two state variables, e.g., θ and π. The derivation follows Dutton's (1976, p. 382). Let φ and Θ be any continuous scalar fields. Then

$$\nabla\frac{D\Theta}{Dt} = \nabla\left(\frac{\partial\Theta}{\partial t} + u\frac{\partial\Theta}{\partial x} + v\frac{\partial\Theta}{\partial y} + w\frac{\partial\Theta}{\partial z}\right) \equiv \frac{D\nabla\Theta}{Dt} + \frac{\partial\Theta}{\partial x}\nabla u + \frac{\partial\Theta}{\partial y}\nabla v + \frac{\partial\Theta}{\partial z}\nabla w. \quad \text{(A1)}$$

By taking the dot product of (A1) with $\varphi\mathbf{w}$ and rearranging terms, we obtain

$$\varphi\mathbf{w}\cdot\frac{D\nabla\Theta}{Dt} \equiv \varphi\mathbf{w}\cdot\left(\nabla\frac{D\Theta}{Dt} - \frac{\partial\Theta}{\partial x}\nabla u - \frac{\partial\Theta}{\partial y}\nabla v - \frac{\partial\Theta}{\partial z}\nabla w\right)$$

$$= \varphi\mathbf{w}\cdot\nabla\left(\frac{D\Theta}{Dt}\right) - \varphi\nabla\Theta\cdot(\mathbf{w}\cdot\nabla)\mathbf{v}. \quad \text{(A2)}$$

The dot product of the vorticity equation (2.12) with $\varphi\nabla\Theta$ gives

$$\varphi\nabla\Theta\cdot\frac{D\mathbf{w}}{Dt} \equiv \varphi\nabla\Theta\cdot(\mathbf{w}\cdot\nabla)\mathbf{v} + \varphi\nabla\Theta\cdot\frac{\alpha}{\alpha_0}\left(\nabla\pi\times c_p\nabla\theta + \nabla\times\mathbf{G}\right). \quad \text{(A3)}$$

Adding (A2) and (A3) produces

$$\varphi \frac{D}{Dt}(\mathbf{w} \cdot \nabla\Theta) \equiv \varphi\mathbf{w} \cdot \nabla \frac{D\Theta}{Dt} + \varphi\nabla\Theta \cdot \frac{\alpha}{\alpha_0}\left(\nabla\pi \times c_p\nabla\theta + \nabla \times \mathbf{G}\right). \tag{A4}$$

Adding $\mathbf{w}\bullet\nabla\Theta D\varphi/Dt$ to both sides provides the identity

$$\frac{D}{Dt}(\mathbf{w} \cdot \varphi\nabla\Theta) \equiv \varphi\mathbf{w} \cdot \nabla \frac{D\Theta}{Dt} + \frac{D\varphi}{Dt}\mathbf{w} \cdot \nabla\Theta + \varphi\nabla\Theta \cdot \frac{\alpha}{\alpha_0}\left(\nabla\pi \times c_p\nabla\theta + \nabla \times \mathbf{G}\right). \tag{A5}$$

When, $\Theta = \theta(\pi, \theta)$, $D\theta/Dt \equiv 0$ and $\mathbf{G} \equiv \mathbf{0}$, (A4) with $\varphi = 1$ implies that $\mathbf{w}\bullet\nabla\theta$ is conserved and remains zero if it is zero initially.

APPENDIX B

Parcel helicity in an unsteady Rankine combined vortex

The unsteady Rankine combined vortex (Davies-Jones and Wood 2006) is an exact axisymmetric solution of the Euler equations of motion and the incompressible continuity equation. We use this solution to illustrate how parcel helicity amplifies in a contracting vortex that is embedded in a convergent flow field. In axisymmetric coordinates (r, φ, z) with wind $\mathbf{v} \equiv (u, v, w)$, the exact solution is

$$u(r) = Dr/Dt = -ar, \tag{B1}$$

$$w(z) = Dz/Dt = 2az, \tag{B2}$$

$$v(r,t) = \begin{cases} M_0\, r/r_c^2(t), r \le r_c(t) \\ M_0/r, r > r_c(t) \end{cases}, \tag{B3}$$

$$\sigma(r,z,t) = \sigma(0,0,t) - \frac{u^2}{2} - \frac{w^2}{2} + \int_0^r \frac{v^2(r',t)}{r'}dr' \tag{B4}$$

where $2a$ is the constant convergence, M_0 is the constant angular momentum outside the vortex core,

$$r_c(t) = r_c(0)\exp(-at). \tag{B5}$$

and σ is the static energy. Since there is no diffusion, the core radius, $r_c(t)$, advects inwards with the flow and parcels do not cross the core wall $r = r_c(t)$. By introducing (B3) into (B4) and performing the integration, we get

$$\sigma(r,z,t) = \sigma(0,0,t) - \frac{u^2}{2} - \frac{w^2}{2} + \begin{cases} M_0^2\, r^2/2r_c^4(t), r \le\ r_c(t), \\ \dfrac{M_0^2}{r_c^2(t)} - \dfrac{M_0^2}{2r^2}, r >\ r_c(t). \end{cases} \quad \text{(B6)}$$

The vorticity

$$\boldsymbol{\omega} = \zeta \mathbf{k} \quad \text{(B7)}$$

where

$$\zeta = \begin{cases} 2M_0/r_c^2(t), r < r_c(t) \\ 0, r > r_c(t) \end{cases} \quad \text{(B8)}$$

by $\zeta = (1/r)\partial(vr)/\partial r$ and (B3).

From (B1) and (B2), trajectories in the (r, z) plane are given by

$$r(\tau) = r_0 \exp(-a\tau), \quad \text{(B9)}$$

$$z(\tau) = z_0 \exp(2a\tau). \quad \text{(B10)}$$

where $r_0 \equiv r(0)$, $z_0 \equiv z(0)$, $\tau \equiv t$ are Lagrangian coordinates. By (B2) and (B10), the radial and vertical velocities of a parcel are

$$u(\tau) = -ar_0 \exp(-a\tau), \quad \text{(B11)}$$

$$w(\tau) = 2az_0 \exp(2a\tau). \quad \text{(B12)}$$

Since parcels in the potential flow outside the core have no vorticity and hence no helicity, we will just consider parcels inside the vortex core. For core parcels, the azimuthal velocity

$$v(\tau) = \frac{M_0 r_0}{r_c^2(0)} \exp(a\tau), \quad \text{(B13)}$$

from (B3) and (B5), the vorticity

$$\boldsymbol{\omega}(\tau) = \zeta(0)\exp(2a\tau)\mathbf{k} \quad \text{(B14)}$$

from (B7), (B8) and (B5), and the helicity

$$h(\tau) = \mathbf{v}(\tau) \cdot \boldsymbol{\omega}(\tau) = 2az_0\zeta(0)\exp(4a\tau) \quad \text{(B15)}$$

from (B12) and (B14). Subtracting the parcel's initial helicity gives

$$h(\tau) - h(0) = 2az_0\zeta(0)[\exp(4a\tau) - 1]. \quad \text{(B16)}$$

Let $L(\tau)$ be a core parcel's kinetic energy minus its static energy. From (B3), (B4), (B6), (B9) and (B10), we obtain

$$L(\tau) = a^2 r_0^2 \exp(-2a\tau) + 4a^2 z_0^2 \exp(4a\tau) + f(\tau) \qquad \text{(B17)}$$

where $f(\tau)$ is a function of τ alone. Let $\Phi(\tau)$ be the Lagrangian integral of L from the initial time to the current time. Then

$$\Phi(\tau) = \frac{a r_0^2}{2}[1 - \exp(-2a\tau)] + a z_0^2[\exp(4a\tau) - 1] + \int_{\tau'=0}^{\tau} f(\tau')d\tau' . \qquad \text{(B18)}$$

Now consider the quantity $\boldsymbol{\omega}\bullet\nabla\Phi$. From (B14) and the chain rule,

$$\boldsymbol{\omega} \cdot \nabla\Phi = \zeta(0)\exp(2a\tau)\frac{\partial\Phi}{\partial z} = \zeta(0)\exp(2a\tau)\left(\frac{\partial\Phi}{\partial r_0}\frac{\partial r_0}{\partial z} + \frac{\partial\Phi}{\partial z_0}\frac{\partial z_0}{\partial z}\right). \qquad \text{(B19)}$$

From (B9) and (B10), $\partial r_0/\partial z = 0$ and $\partial z_0/\partial z = \exp(-2a\tau)$, and from $\partial/\partial z_0$ of (B18),

$$\frac{\partial\Phi}{\partial z_0} = 2a z_0[\exp(4a\tau) - 1]. \qquad \text{(B20)}$$

Therefore, (B19) becomes

$$\boldsymbol{\omega} \cdot \nabla\Phi = 2a z_0 \zeta(0)[\exp(4a\tau) - 1]. \qquad \text{(B21)}$$

Comparison of (B16) and (B21) shows that

$$h(\tau) - h(0) = \boldsymbol{\omega} \cdot \nabla\Phi. \qquad \text{(B22)}$$

Thus, for a contracting Rankine combined vortex, gain in parcel helicity is equal to the scalar product of vorticity with the gradient of the Lagrangian integral of the kinetic energy of a parcel minus its static energy. This same relationship is true for any inviscid, homentropic (constant θ) motion [see (3.26) in Part I].

We now express the helicity gain in terms of Lagrangian coordinates. From (B14) evaluated initially and (B20) and (B22), we find that

$$h(\tau) - h(0) = \zeta(0)\frac{\partial\Phi}{\partial z_0} \qquad \text{(B23)}$$

where $\zeta(0)$ is the only nonzero contravariant component of barotropic vorticity and $\partial\Phi/\partial z_0$ is the corresponding covariant component of $\nabla\Phi$.

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